SEMSTRAL EXAMINATION
M. MATH I Year, I SEMESTER
2005-2006
COMPLEX ANALYSIS

Max. Marks:100 DURATION: 3hrs

1. If f is the sum of a convergent power series on a disk $D(0,R)$ prove that the integral of f over any closed path γ in $D(0,R)$ is zero. [DO NOT USE CAUCHY's THEOREM IN ANY FORM!] [15]

2. If K is a compact set in $\mathbb{C}$ show that $\mathbb{C}\backslash K$ is connected if and only if $\mathbb{C}^*\backslash K$ is connected. [15]

3. Prove that the metric space $H(\Omega)$ is complete (with the metric defined in class) for any region Ω. If $\Omega = \{z : \mathrm{Re}(z) > 0\}$, prove that $H(\Omega)$ is also separable. [15]

4. Prove using contour integration that $\int\limits_0^\infty \frac{\log(x)}{(1+x^2)^2} dx = -\pi/4$. [You may use the contour displayed on the black board] [15]

5. Prove that $\{e^{1/z} : 0 < |z| < \varepsilon\} = \mathbb{C}\backslash\{0\}$ for any $\varepsilon > 0$. [10]

6. Let u be a harmonic function (i.e. a twice continuously differentiable function with $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$). It can be shown that there is a harmonic function v with $u + iv$ analytic (on $\mathbb{C}$, as a function of $x + iy$). Using this prove The Mean Value theorem for harmaonic functions:

$u(a,b) = \int\limits_0^1 u(a + R\cos(2\pi it), b + R\sin(2\pi it))dt$ for any $(a,b) \in \mathbb{R}^2$ and any $R > 0$. [10]

7. Prove that $\phi(z) = \frac{z-2}{2z+i}$ maps $\{z : \left|z - \frac{1}{2}\right| = \frac{1}{2}\}$ onto a circle. [No need to find the centre and radius]. [10]

8. Let $\Omega = \{z : \mathrm{Re}(z) > 0\}$. If f is continuous on $\bar{\Omega}$ and holomorphic on Ω and if $|f(z)| \leq 1$ for all z on the bondary of Ω does it follow that $|f(z)| \leq 1$ for all $z \in \Omega$?. Justify your answer. [10]